EVOLUTION OF NATURAL CONVECTION IN A TALL RECTANGULAR CAVITY HEATED FROM BELOW

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ABSTRACT

A study is made of free convection in a vertical cavity whose side walls are assumed adiabatic while the bottom and the top are respectively heated and cooled by a uniform heat flux. The problem is formulated in terms of Navier-Stokes and energy equations. A stability analysis is first presented to set the stage for the derivation of the evolution equations of the weakly nonlinear velocity and temperature fields. Furthermore, solutions of the fully nonlinear set of governing equations are obtained to show the development of multicellular flows at high Rayleigh numbers.

Key words : Natural convection, vertical cavity, amplitude equation, multicellular flows.

NOMENCLATURE

H'	height of cavity, m
H	aspect ratio (H'/L')
k	thermal conductivity, W/mK
L'	width of cavity, m
Pr	Prandtl number
R	Rayleigh number based on L'
Ra	Rayleigh number based on H'
T	temperature, K
$\vec{u}$	fluid velocity, m/s
x	horizontal coordinate, m
y	vertical coordinate, m
a	thermal diffusivity, m^2/s
b	thermal expansion coefficient
ΔT	temperature scale, K
n	kinematic viscosity, m^2/s
Ψ	stream function (dimensionless)
q	deviation of dimensionless temperature from conduction state

INTRODUCTION

Natural convection flows in tall vertical cavities have been extensively investigated during the last fifty years. All theses studies are focused on the case of differentially heated cavities, with the vertical side walls being maintained at two different temperatures while the horizontal end walls are assumed adiabatic. Depending on the aspect ratio, the Rayleigh number and the Prandtl number, various flow regimes have been obtained , both numerically and experimentally. For very tall cavities and small Rayleigh numbers, the end effects are of limited extend and the flow is almost parallel in the core region [1] (Batchelor 1954). As the Rayleigh number is increased, the flow may become unstable and the well-known cats eyes may develop [2, 3, 4] (Hart 1971, Korpela et al 1973, Thangam and Chen 1986). In cavities with finite aspect ratios, it was found that the flow is still parallel in the core region, but there also develops a thermal stratification. As the Rayleigh number is increased beyond a certain critical value, this so-called

buoyancy -boundary layer flow becomes unstable against either steady or traveling waves, depending on the cavity aspect ratio and the fluid Prandtl number [5, 6, 7, 8] (Gill 1966, Gill and Davey 1969, Bergholz 1978, Roux et al 1980).
While differentially heated cavities have been extensively investigated, very few studies were devoted to tall vertical cavities heated from below. In fact, this problem has been considered by Normand 1984 for the case of a vertical cavity [9] and by Cessi and Young 1992 for the case of a tilted slot [10]. Both studies were focused on the derivation of the amplitude equations to describe the spatio-temporal evolution of weakly nonlinear flows and their stability in very thin slots.
In this study, we consider the case of a tall cavity heated from below by a uniform heat flux. Numerical solutions will be shown for a wide range of Rayleigh number. The obtained results will be interpreted in terms of the long-wave theory.

MATHEMATICAL ANALYSIS

Consider a fluid confined in a tall vertical slot with a constant vertical temperature gradient due to heating from below. The aspect ratio is sufficiently large for the effects of the end walls to be of limited extend and the flow to be almost parallel in the core region.
The fluid is considered incompressible with constant thermophysical properties, except for the temperature-dependent density in the buoyancy force. The vertical walls are assumed adiabatic, while the top and bottom boundaries are subject to the fixed-flux condition.
The velocity and temperature fields are therefore governed by the following Navier-Stokes and energy equations:

$$w_t - \Pr \nabla^2 w - \Pr R q_x = w_y \Psi_x - w_x \Psi_y \qquad (1)$$

$$q_t + y_x - \nabla^2 q = q_y y_x - q_x y_y \qquad (2)$$

with $w = -\nabla^2 \Psi$, Ψ being the stream function, and q the deviation of the temperature field from the conduction state, i.e.

$$u = y_y \, , v = -y_x \, , T = -y + q$$

where u, v, T are the dimensionless velocities and temperature.

All quantities are made dimensionless using L', L'2/ α, and α/L' as length, time and velocity scales. The problem is governed by three parameters, namely, the aspect ratio H = H'/L', the Prandtl number Pr and the Rayleigh number R = gβΔTL'/να where g, β, ν, α are the gravitational acceleration, volumetric expansion coefficient, kinematic viscosity and thermal diffusivity. Note that ΔT is here defined as bL' with b being the imposed vertical temperature gradient.
The thermal boundary conditions to be satisfied in the present problem are therefore

$$q_x = 0 \quad at \; x = \pm 1/2$$
$$q_y = 0 \quad at \; y = \pm H/2 \qquad (3)$$

The above equations are solved by the control volume approach with second-order central scheme for spatial derivatives, and first-order backward (implicit) scheme for time derivatives. The typical mesh size is 21X81 for a cavity of aspect ratio 8. The typical time step is 0.005.

Linear stability analysis

As the cavity is subject to a temperature gradient opposite to gravity, we are dealing with a top-heavy configuration. The motionless conduction state is therefore potentially unstable and convection will arise as soon as buoyancy is strong enough to overcome viscous forces. The critical condition for the onset of convection may be determined by the linear stability theory: For small perturbations of the conduction state, we may neglect the nonlinear terms in the governing equations for ψ and θ to obtain the linear system.

$$w_t - \Pr\nabla^2 w - \Pr R q_x = 0 \tag{4}$$

$$q_t + y_x - \nabla^2 q = 0 \tag{5}$$

subject to the homogeneous boundary conditions (3). For mathematical simplicity, we will consider slip condition at the vertical walls.
For *H>>1*, this eigenvalue problem is readily solved by

$$y = F(t, y)\cos p x \tag{6}$$

$$f = G(t, y)\sin p x \tag{7}$$

with a critical Rayleigh number

$$R_c = p^4\left(1+\frac{n^2}{H^2}\right)^3 \tag{8}$$

where $n=1,2,3,...$ is the vertical mode number of the eigenfunctions ($n=1$ corresponds to a unicellular flow, $n=2$ corresponds to a superposed bicellular flow, etc.).
Equation (8) shows that in the limit *H>>1*, the critical Rayleigh number does not depend on the type of thermal boundary conditions imposed on the horizontal walls, i.e. it is the same under fixed-flux or fixed-temperature heating. The above result also shows that multiple multicellular flows may arise at Rayleigh numbers close to $R_0 = p^4$. For example, in a cavity of aspect ratio *H = 10*, the critical Rayleigh number is $R_c \approx 1.03p^4$ for the unicellular flow and $R_c \approx 1.12p^4$ for the bicellular flow.
To study the development of the flow at supercritical Rayleigh numbers we need to take into account the nonlinear interactions as shown in the next section.

Nonlinear evolution

In large aspect ratio cavities, we are dealing with nonlinear interactions between long waves. To capture the vertical structure of the solutions we therefore introduce a length scale $x = ey$, a time scale $t = e^2 t$ and a supercritical Rayleigh number $R \approx R_0 + e^2 R_2$ where $e \equiv \frac{1}{H}$. We then look for solutions as power series of e, i.e.

$$y = e f_1 + e^3 f_3 + ... \tag{9}$$

$$q = e q_1 + e^3 q_3 + ... \tag{10}$$

By substituting these series in Eqs. (1-2) we obtain a hierarchy of equations to be solved order by order.

At leading order we have

$$f_{1xxxx} - R q_{1x} = 0 \tag{11}$$

$$q_{1xx} - f_{1x} = 0 \tag{12}$$

whose solutions under the boundary conditions

$$f_1 = q_{1x} = 0 \quad at \quad x = \pm\frac{1}{2} \tag{13}$$

are

$$f_1 = p f(t, x)\cos px \tag{14}$$

$$q_1 = f(t, x)\sin p x + g(x, t) \tag{15}$$

where f and g are functions of the slow variables t and x to be determined at the next order. At order e^3, we have to deal with an inhomogeneous system of equations. The solvability condition then requires that the inhomogeneous terms be orthogonal to the eigenfunctions of the homogeneous system, leading to the following equations for f and g

$$pf_t = R_2 f + 3 f_{xx} - R_o f g_x \tag{16}$$

and

$$g_t = g_{xx} - ff_x \tag{17}$$

which are the amplitude equations governing the spatio-temporal evolution of the velocity and temperature fields. It can be easily deduced from these equations that the system can reach either a unicellular or multicellular steady state, but prediction of the preferred mode is an open question .

Dispersion Relation

At early times, small perturbations will grow according to the linear equation

$$pf_t = R_2 f + 3 f_{xx} \tag{18}$$

from which we deduce the dispersion relation

$$p l = R_2 - \frac{3n^2 p^4}{H^2} \tag{19}$$

where l is the growth rate and n the mode number, $p = 1 + \mathrm{Pr}^{-1}$

We note from this relation that the unicellular flow *(n=1)* is the mode having the highest growth rate and the lowest critical Rayleigh number.

Parallel Flow Approximation

For steady state flow in the central region, we may neglect both space and time derivatives in the above equations to get

$$g_t = \frac{1}{2} f^2 \tag{20}$$

$$f = \left(2R_2 / R_0\right)^{1/2} \tag{21}$$

We thus recover the solution obtained from the parallel-flow approximation

NUMERICAL SOLUTIONS

Fig. 1 shows the stream lines and isotherms patterns in a cavity under fixed-flux heating condition with an aspect ratio H=8, a Prandtl number Pr = 0.7 and a Rayleigh number Ra=2 500 000 (note that hereafter, Ra is the Rayleigh number based on the height H'of the cavity). This figure clearly shows that the unicellular flow is parallel in the core region. It should be noted that for smaller Ra, the flow becomes weaker, and eventually vanishes if Ra is decreased below a critical value as predicted by the linear stability theory.

Fig1.I.C=Zero,
Ra=2.5E6, Dt=0.01

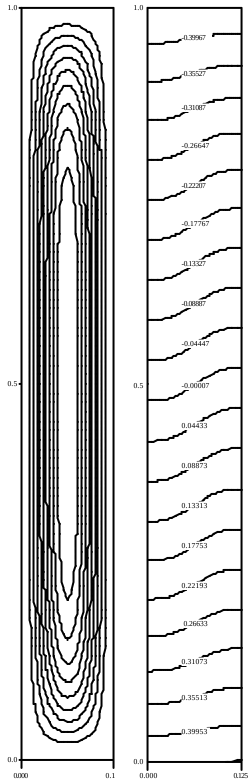
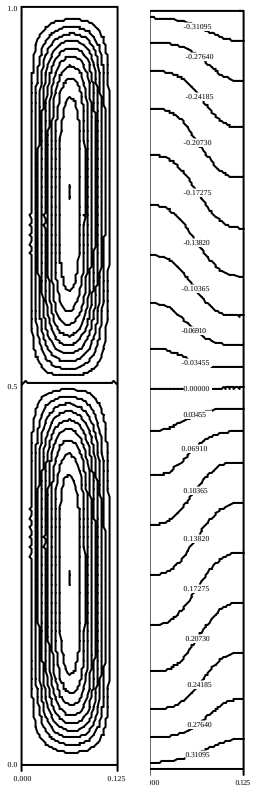


Fig2. I.C=Zero,
Ra=4.0E6, Dt=0.01



For Ra= 4 000 000, the solution obtained by using the solution at Ra=2 500 000 as initial condition (I.C.) is a unicellular flow (very similar to that of Fig.1). By starting with a zero-initial condition (Ψ = T = 0), we obtained a symmetrical bicellular flow, as shown in Fig.2. In other words, at Ra=4 000 000, we obtain two different solutions, depending on the I.C. It should be noted that for smaller Rayleigh numbers (below 3 000 000) there exists only one solution, namely the unicellular flow, independently of the I.C.

Fig3a. I.C=Zero,
Ra=6.0E6, Dt=0.001

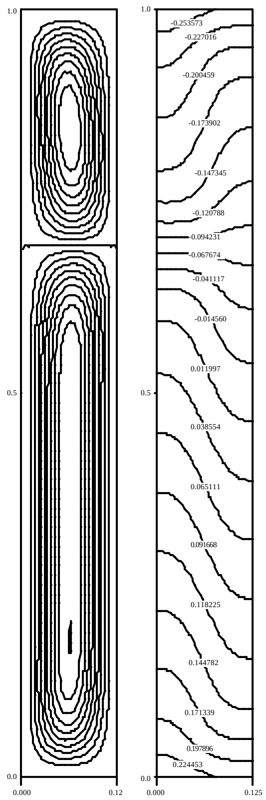


Fig3b. I.C=Zero,
Ra=6.0E6, Dt=0.01

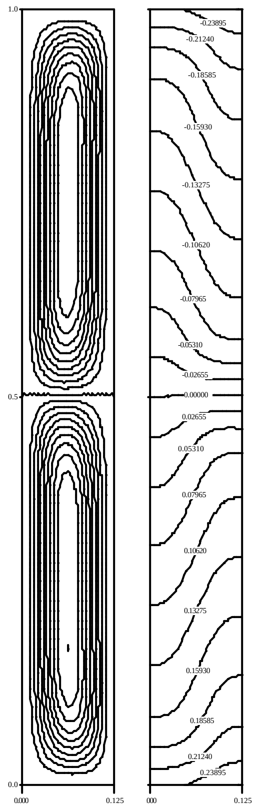


Fig.3 shows the solutions for Ra= 6 000 000. By starting with a zero initial condition, we may obtain two different flow patterns, depending on the chosen time step: The symmetric bicellular flow in Fig3a was obtained using a time step dt = 0.01 while the non-symmetric bicellular flow shown in Fig 3b was obtained with a time step dt = 0.001. Multisolutions were also found at Ra = 15 000 000: One, two and three cells flow patterns shown in Fig4a,b,c were in fact obtained using different I.C. and time steps.
These results confirm the prediction of the stability analysis that multisolution may exist at supercritical Rayleigh numbers. Which one is the preferred mode at a given Rayleigh number is still an open question.

CONCLUSION

A linear stability analysis, a nonlinear evolution theory and a numerical simulation of convection were presented to describe the development of parallel flows in a tall vertical cavity heated from below. The novelty of this work lies in its thin geometry, in the multiplicity of its solutions and in the multicellular structure of the developed flows.

Fig4a. I.C=1cell weak
Ra=1.5E7, Dt=0.01

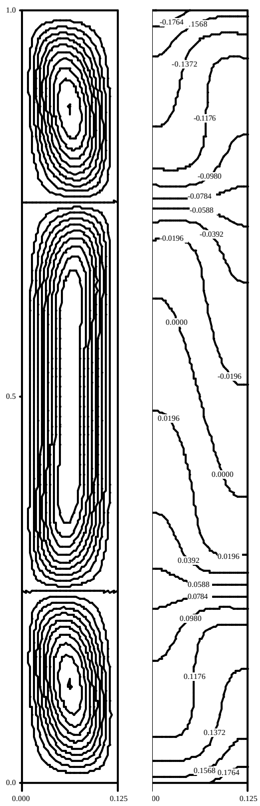


Fig4b. I.C=Zero
Ra=1.5E7, Dt=0.01

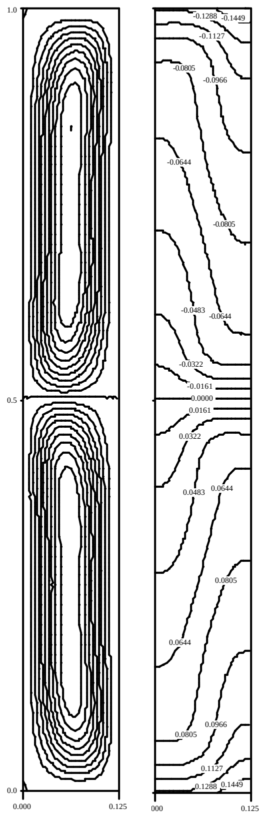
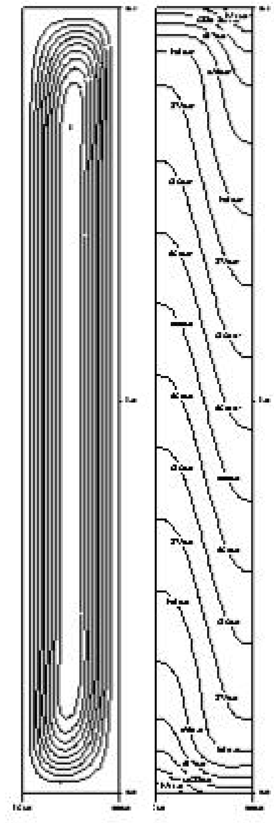


Fig4c. I.C=1cell strong
Ra=1.5E7, Dt=0.01



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