EVOLUTION OF CONVECTION FLOW IN A HORIZONTAL CAVITY

T. Hung Nguyen, P. Bournot*, G. LePalec* and M. Prud'homme
Département de génie mécanique, École Polytechnique de Montréal, Canada
and
*Institut de Mécanique de Marseille, Université de la Méditerranée
Marseille, France
Email : the-hung.nguyen@polymtl.ca, Fax : (514) 340 5917

ABSRACT

A study is made of convection flow in a horizontal cavity whose boundaries are subject to a uniform heat flux. Equations for three-dimensional perturbations are solved to determine the critical Rayleigh number and wavenumber as functions of the Prandtl number. It is found that the onset of instabilities corresponds to a Hopf bifurcation of the base flow. Two-dimensional longitudinal perturbations are the most unstable for Prandtl numbers smaller than 4.2. For higher Pr, the most unstable perturbations are three-dimensional. Numerical solutions of the fully nonlinear governing equations are presented for a wide range of Rayleigh numbers to show the transition of flows from steady to oscillating regimes.
Key words: Fixed-flux convection, parallel flow, nonlinear stability, oscillating flow regime.

NOMENCLATURE

A	aspect ratio (L/H)
H	height of cavity, m
k	thermal conductivity, W/mK
k	wave number in x-direction, m^{-1}
l	wave number in z-direction, m^{-1}
L	width of cavity, m
Pr	Prandtl number
Ra	Rayleigh number
T	temperature, K
$\vec{u}$	fluid velocity, m/s
x, z	horizontal coordinates, m
y	vertical coordinate, m
a	temperature gradient in y-direction
a_T	thermal diffusivity, m^2/s
b_T	thermal expansion coefficient, K^{-1}
b	direction of propagation of perturbations, rad
ΔT	temperature scale, K
n	kinematic viscosity, m^2/s
Ψ	stream function (dimensionless)
s	complex amplification rate, s^{-1}
q	deviation of dimensionless temperature from convection state

INTRODUCTION

It is well known that unicellular flows in rectangular cavities may become unstable at high Rayleigh numbers [1, 2, 3, 4]. Instabilities in a vertical cavity with a fixed temperature difference on the side walls were examined by Bergholz [5], Korpela et al. [6] for various levels of stratification. Suslov and Paolucci [7] extended the analysis to non-Boussinesq conditions without stratification of the base temperature field. It was found that transition from stationary to oscillatory regime was possible, and that the oscillations were either of hydrodynamic or thermal origin, depending on the Prandtl number. Nield [8] showed that different instability regimes could be obtained in a horizontal porous layer submitted to an oblique temperature gradient, by varying the vertical to lateral temperature gradient ratio. Kimura et al. [9] considered the stability of a porous layer heated from below by a constant heat flux. A critical transition from the steady parallel base flow to an oscillatory regime was found.
In this paper we consider the evolution and stability of convection flows in a horizontal cavity whose boundaries are subject to a uniform heat flux. Equations for three-dimensional perturbations are solved to determine the critical Rayleigh number and critical wavenumber as functions of the Prandtl number. Numerical solutions of the governing equations are obtained to show the transition from steady to oscillating flow regimes.

MATHEMATICAL ANALYSIS

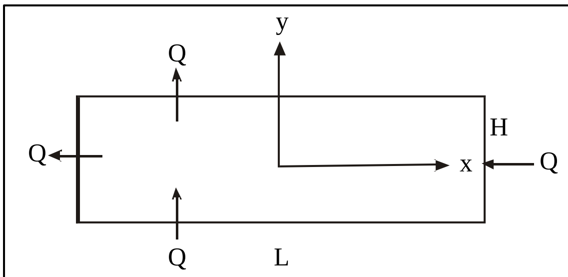

FIG. 1

Geometry and boundary conditions

Let us consider the convection flow in a thin horizontal rectangular cavity that is heated by uniform heat fluxes, as shown in Fig.1. It is assumed that the temperature differences are small enough to ensure the validity of the Boussinesq approximation. For a fluid of constant kinematic viscosity ν, thermal diffusivity α_T and coefficient of thermal expansion β_T, we may define the length, temperature, velocity and time scales defined as

$$H\ ,\ \Delta T = QH/k\ ,\ a_T/H\ ,\ H^2/a_T \qquad (1)$$

with all fluid properties taken as constants evaluated at some reference temperature T_0.

Within the Boussinesq approximation $\rho = \rho_0[1 - \beta_T(T - T_0)]$ for the buoyancy force , the dimensionless governing equations may be expressed as

$$\nabla \cdot \vec{\mathbf{u}} = 0 \qquad (2)$$

$$\frac{D}{Dt}\vec{\mathbf{u}} = -\nabla p + \Pr \nabla^2 \vec{\mathbf{u}} - \mathrm{Ra}\Pr T\hat{\mathbf{g}} \qquad (3)$$

$$\frac{DT}{Dt} = \nabla^2 T \qquad (4)$$

with the Prandtl number $\Pr = \nu/\alpha_T$ and the Rayleigh number $\mathrm{Ra} = g\beta_T \Delta T H^3/\nu\alpha_T$.

Base Flow

It is well known that for sufficiently large values of the aspect ratio $A = L/H$, a parallel flow is developed in the central region of the cavity. Consequently, this base flow can be described by a stream function $\psi = \psi(y)$. The continuity Eq.(2) is then automatically verified, while the momentum Eq.(3) simplifies to

$$\psi'''' - \mathrm{Ra}\frac{\partial T}{\partial x} = 0 \qquad (5)$$

subject to the no-slip boundary conditions $\psi = \psi' = 0, \ \ y = \pm 1/2$.

In conjunction with Eq.(4) under the fixed-flux boundary conditions, it may be readily shown that the stream function and temperature are

$$\psi(y) = \frac{\alpha\,\mathrm{Ra}}{4!}\left(y^2 - \frac{1}{4}\right)^2 \tag{6}$$

$$T = \alpha x - y + \frac{\alpha^2\,\mathrm{Ra}}{4!}\left(\frac{y^5}{5} - \frac{y^3}{6} + \frac{y}{16}\right) \tag{7}$$

where the horizontal temperature gradient α may be determined by integrating Eq.(4) in conservative form over an arbitrary section of the cavity, using the boundary condition $\partial T/\partial x = 1$ on the side walls. We then obtain

$$\frac{\alpha^3\,\mathrm{Ra}^2}{9!} + \alpha\left(1 - \frac{\mathrm{Ra}}{6!}\right) - 1 = 0 \tag{8}$$

for α as a function of the Rayleigh number.

Linear Stability Analysis

Let small perturbations of the temperature, pressure, and velocity fields of the form

$$\tilde{f} = \mathrm{Re}\left\{\hat{f}(y)\right\}\exp\left[\sigma t + i(kx + lz)\right] \tag{9}$$

be added to the base flow, where $\tilde{f}$ is a complex quantity, k and l are the real longitudinal and transverse wavenumbers, respectively, and $\sigma = \sigma_r + i\sigma_i$ is the complex amplification rate.

The set of governing equations (2)-(4) may be linearized to first order in small quantities to obtain the eigenvalue problem

$$\begin{bmatrix} \tilde{k}^2(ik\psi' - \Pr L) & k(k\psi' + i\Pr L)D + l^2\psi'' & 0 \\ 0 & \Pr L^2 + ik(\psi''' - \psi' L) & -\tilde{k}^2 \mathrm{Ra}\Pr \\ -\alpha & -g' & L - ik\psi' \end{bmatrix} \begin{Bmatrix} \hat{u} \\ \hat{v} \\ \hat{T} \end{Bmatrix} = \sigma \begin{bmatrix} -\tilde{k}^2 & ikD & 0 \\ 0 & L & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{u} \\ \hat{v} \\ \hat{T} \end{Bmatrix} \tag{10}$$

where $\tilde{k}^2 = k^2 + l^2$, D = d/dy , L represents the operator $D^2 - k^2 - l^2$ and the prime designates derivative with respect to y.

Solutions to this problem can be achieved as follows. The various terms in Eq.(10) are discretized according to central difference schemes. The eigenvalues of the resulting discrete matrix are then computed from a standard IMSL subroutine such as EIGENC or QZ. The eigenvalue problem is solved holding k, l, Pr constant each time. Newton's method is used to find the value of Ra for which the maximum growth rate σ_r cancels. Repeating the procedure for different wavenumbers k and l determines a marginal stability surface for a given Prandtl number Pr. The minimum value of Ra on the surface corresponds to the critical Rayleigh number Ra_c.

Let introduce the angle β to define the direction of propagation of the disturbances in such a way that $k = \tilde{k}\cos b$ and $l = \tilde{k}\sin b$. A detailed analysis of this problem [4] showed that 2-dimensional longitudinal perturbations are the most unstable for Prandtl numbers smaller than 4.2. For higher Pr values, the most unstable perturbations are 3-dimensional with a transverse wavenumber becoming more important as the Prandtl number increases. Fig. 2 displays the marginal stability curves for different values of Pr. Solutions of the eigenvalue problem Eq.(10) reveal that σ_i is different from zero at any Prandtl number, i.e. the instabilities are always oscillating. This result indicates a Hopf bifurcation of the base flow, for which the disturbances are characterized by oscillations in amplitude, unlike what is found in the case of a vertical cavity with a fixed temperature difference imposed between the walls [6].

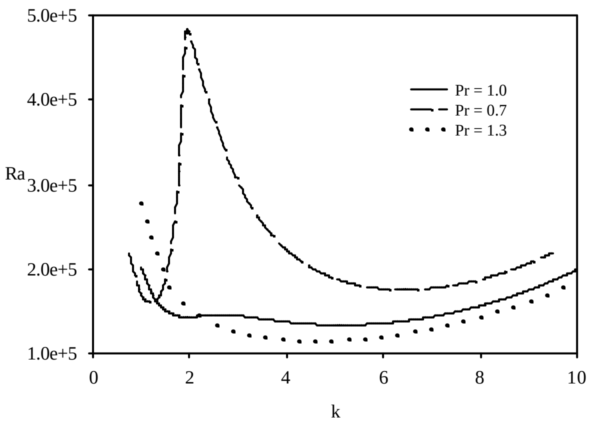

Fig. 2 : Marginal stability for l = 0
Pr = 0.7 and 1.0

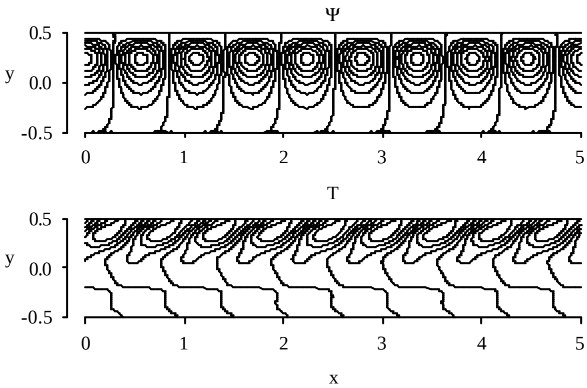

Fig. 3 : Critical disturbances along x
Pr = 1.0, Ra_c = 133200, k_c = 5.12

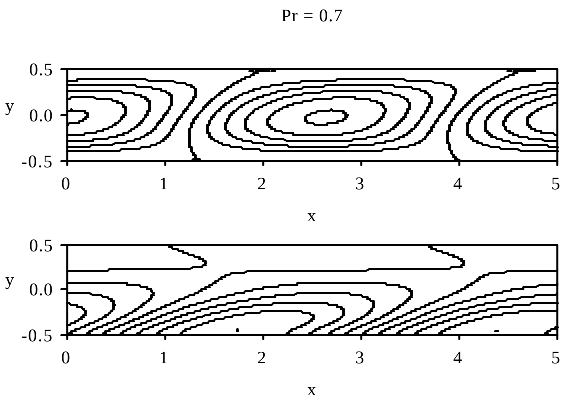

Fig. 4: Critical disturbances for l = 0, Ra_c = 159850, k_c = 1.188, σ_I = 68.477

It should be noted from Fig. 2 that the stability curves for Pr = 0.7 and 1.0 display two relative minima. Each minimum corresponds to the critical Rayleigh number for a different mode of instability. For Pr = 1.0, the most unstable instability mode is the one associated with the relative minimum at the higher wavenumber on the marginal stability curve. The corresponding disturbance streamlines and isotherms are shown in Fig. 3. For Pr = 0.7, the most unstable mode is located at the lower wavenumber and the distinction between the branches of the stability curve corresponding to each instability mode is very clear on the plot. Figure 4 depicts the streamlines and isotherms of perturbations at the critical Rayleigh number.
The above results show that the onset and development of the instabilities are strongly influenced by the Prandtl number. It can be reasonably expected that instabilities are more likely to be shear-driven for Pr < 1, as thermal

disturbances of the base flow are then prone to be dissipated faster than hydrodynamic disturbances due to the larger molecular heat diffusion. For Pr >1, viscous diffusion is greater than heat diffusion, and instabilities are expected to be buoyancy-driven. In fact, a detailed analysis of the kinetic and potential energies at the onset of instabilities showed that they are of thermal origin when Pr $\geq$ 0.9, but shear-driven otherwise [4].

NUMERICAL SOLUTIONS

The fully nonlinear Navier-Stokes and energy equations are numerically solved by the control volume method for a cavity of aspect ratio 8. The grid size is 21X81, with a time step varying from 0.01 to 0.0001, depending on the Rayleigh number. Some typical results are shown below for a fixed Prandtl number Pr = 0.7 and a Rayleigh number varying from 100 000 to 200 000. Fig. 5 shows the steady parallel flow obtained at Ra= 120 000. Fig. 6 shows the flow obtained at Ra = 160 000 which is still a parallel unicellular flow, but its amplitude weakly oscillates in time as can be seen in Fig. 9. The flow at Ra = 180 000 is shown in Fig. 7. Solutions obtained at Ra = 200 000 are presented in Fig. 8 and 10.

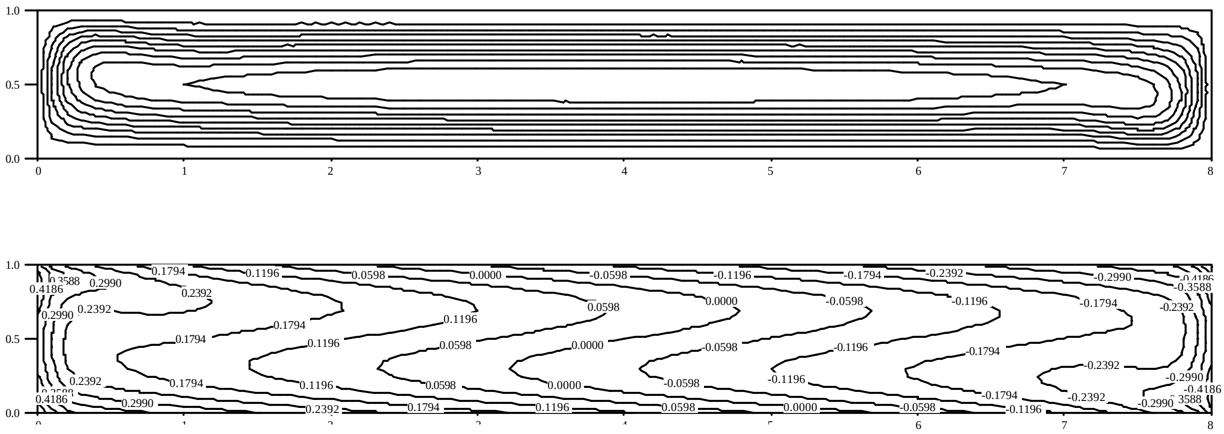

Fig 5 : Ra=1.2E5, Dt=0.01, I.C=Zero, A=8

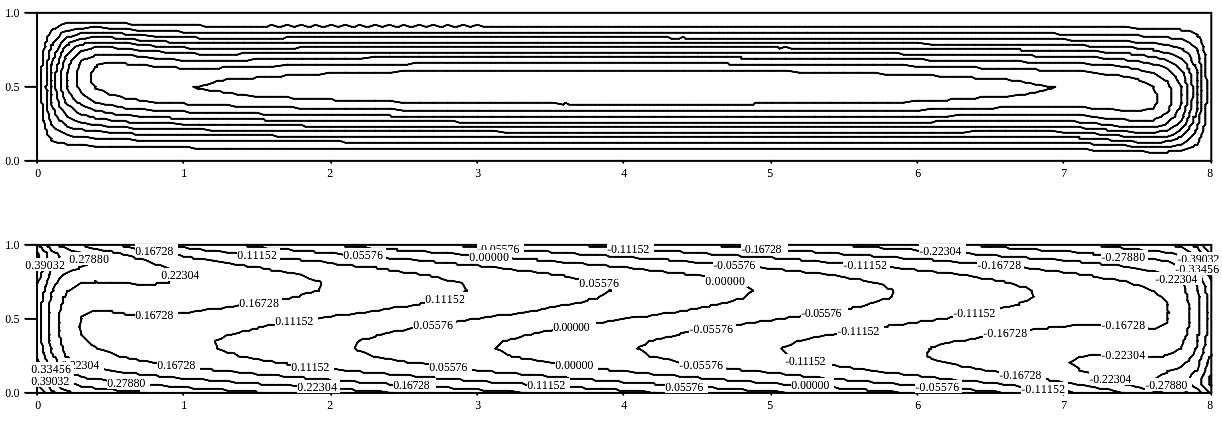

Fig 6 : Ra=1.6E5, Dt=0.01, I.C=Zero, A=8

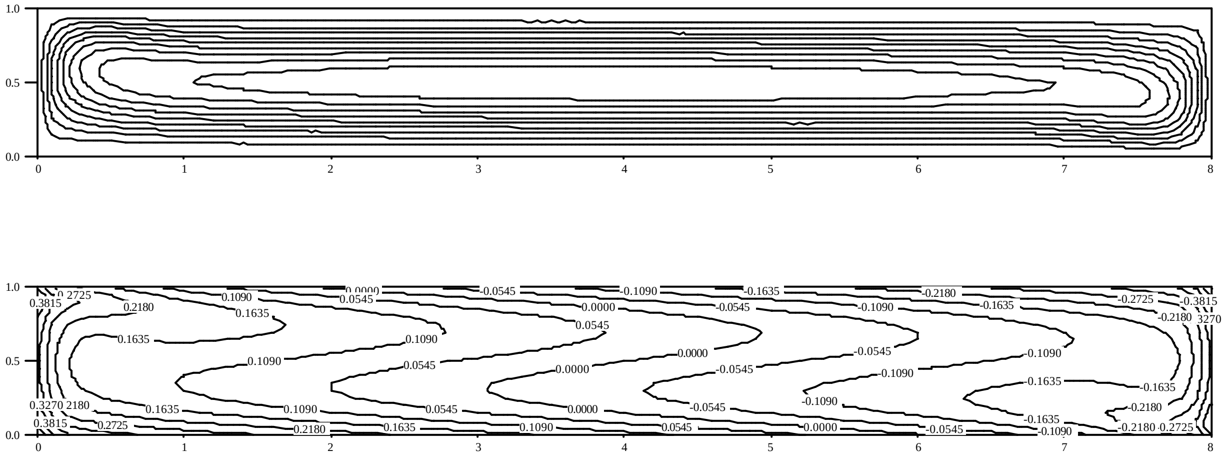

Fig 7 : Ra=1.8E5, Dt=0.01, I.C=Zero, A=8

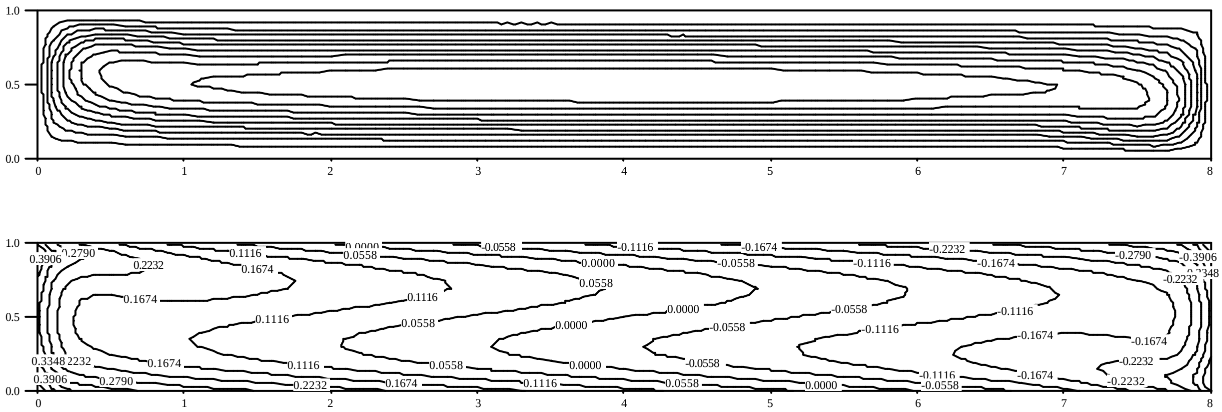

Fig 8 : Ra=2.0E5, Dt=0.01, I.C=Zero, A=8

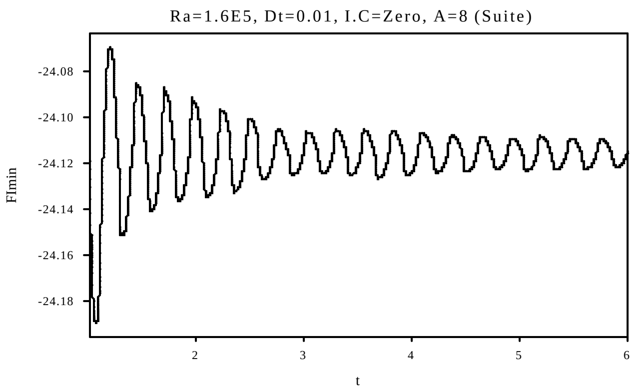

Fig. 9: Minimum stream function vs time

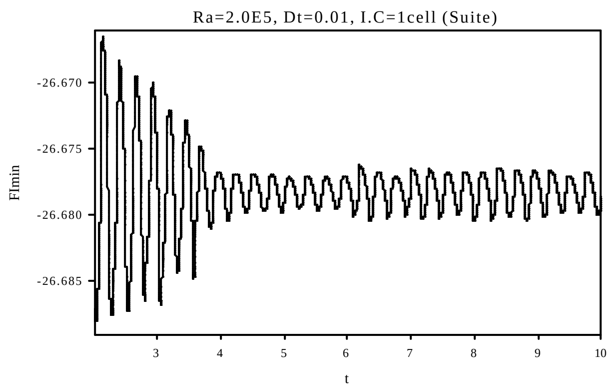

Fig. 10: Minimum stream function vs time

It can be verified from the stream lines patterns that the flow is parallel for the whole range of Rayleigh numbers considered here. More importantly, it was found that a steady state solution is obtained, and the flow is stable, for Rayleigh numbers less than 160 000. At higher Rayleigh numbers, the flow remains essentially parallel, but its amplitude oscillates in time, albeit rather weakly. The apparition of oscillations at Ra = 160 000 is an important result as it clearly confirms the stability analysis which predicts a Hopf bifurcation of the parallel flow at a critical Rayleigh number Ra = 159 850 as shown in Fig. 4.

CONCLUSION

Natural convection in a horizontal cavity under the fixed-flux boundary condition was shown to develop into essentially parallel flows. These flows were shown to become unstable against either 2D or 3-D oscillating perturbations, depending on the Prandtl number. The stability analysis predicts that the onset of instabilities corresponds to a Hopf bifurcation at a critical Rayleigh number Ra = 159 850 for Pr = 0.7. A direct simulation of the Navier-Stokes equations reveals indeed a weakly oscillating parallel flow at Ra = 160 000. These results warrant the development of a nonlinear long-wave theory to study the spatio-temporal evolution of convection flows under fixed-flux conditions.

ACKNOWLEDGEMENT

This study was partly done while THN was on sabbatical leave at the Institut de Mécanique de l'Université de la Méditerranée. The hospitality of Prof. D. Dufresne, Director of IM2, is gratefully acknowledged.
This work was supported by the National Sciences and Engineering Research Council of Canada under grant OGP 626.

REFERENCES

1. Bejan A., *Int. J. Heat Mass Transfer* (1983), vol. 26, pp. 1339-1346.

2. Nguyen T.H., Nguyen D.L. and Lavoie S., Convective Heat Transfer in a Vertical Annular Porous Layer Exposed to a Constant Heat Flux, *ASME HTD* (1986), vol.84, pp. 105-115.

3. Lavoie S., Nguyen T.H. and Laberge C.A., Heat Transfer by Natural Convection Between Two Concentric Cylinders, *Proc. 5th Int. Conf. Num. Methods in Thermal Problems, Montreal* (1987), vol. V, part 1, pp. 536-546.

4. Prud'homme M , Nguyen T.H and Bougherara H., Stability of Convection Flow in a Horizontal Fluid Layer Heated by Uniform Heat Fluxes, Int. Comm. Heat Mass Transfer (2003) in press.

5. Bergholz R.F., Instability of Steady Natural Convection in a Vertical Fluid Layer, *J. Fluid Mech*. (1978), vol.84, pp. 743-768.

6. Korpela S.A., Gözüm D. and Baxi C.B.¸ On the Stability of the Conduction Regime of Natural Convection in a Vertical Slot, *Int. J. Heat Mass Transfer* (1973, vol. 15, pp. 1683-1690.

7. Suslov S.A. and Paolucci S., *Int. J. Heat Mass Transfer* (1995), vol. 38, pp. 2143-2157.

8. Nield D.A., *Int. J. Heat Mass Transfer* (1991), vol. 34, pp. 87-92.

9. Kimura S., Vynnycky S. and Alavyoon F., *J. Fluid Mech*. (1995), vol. 294, pp. 231-257.