METHOD FOR SOLVING THE SIDEWAY HEAT EQUATION APPLICATION TO THE NON-INTRUSIVE TEMPERATURE MEASUREMENT

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RESUME

In the present study, an inverse heat conduction problem is considered, the Sideways Heat Equation, which is a model of a problem, where one wants to determine the temperature on both sides of a thick wall, but where one side is inaccessible to measurements. Considering the problem of an intrinsic thermocouple attached to a slab, and no thermal contact, a transfer function is proposed. It is based on Duhamel Integral and the measured response is approximated by a quasi-periodical function. The derived transfer function is the product of two transfer functions, for the slab and for the intrinsic thermocouple respectively. To validate the present model, a comparison with actual measurements is carried out. It was found that the present model can predict the inner temperatures more accurately than the previous model, which did not take into account the effect of the thermocouple.

Key words : Sideways Heat Equation, Duhamel Integral, Thermocouple, Non-Intrusive temperature measurement, Transfer function

NOMENCLATURE

A_n	dimensionless constant in Eq. (11)
Bi	Biot number = $h \cdot L/\lambda_S$
B_1	dimensionless constant in Eq. (15)
B_2	dimensionless constant in Eq. (16)
$C_2(\omega t)$	Fresnel Integral in Eq. (22)
D	parameter = $B_2\sqrt{\alpha_S/r_w^2}$, ($sec^{-1/2}$)
f	frequency, (Hz)
h	fluid-slab heat transfer coef., ($W \cdot m^{-2} \cdot K^{-1}$)
$K_n(\xi)$	dimensionless constant in Eq. (8)
L	slab thickness, (m)
M_{S1}	dimensionless constant in Eq. (10)
M_{S2}	dimensionless constant in Eq. (10)
M_{T1}	dimensionless constant in Eqs. (18)
M_{T2}	dimensionless constant in Eqs. (19)
NFo	Fourier number = $\alpha_S t/L^2$
r_w	thermocouple radius, (m)
$S_2(\omega t)$	Fresnel Integral in Eq. (22)
t	time, (sec.)
t^*	dimensionless time = $\alpha_S t/r_w^2$
T	temperature, (K)
x	position in the slab, (m)

Greek symbols

α	thermal diffusivity, ($m^2 \cdot sec^{-1}$)
β	dimensionless constant = $\dfrac{\lambda_T/\lambda_S}{\sqrt{\alpha_T/\alpha_S}}$
δ_n	n^{th} root of Eq. (6)
ϕ	phase lag of the response, (rad)
η	dimensionless constant = $\omega \cdot L^2/\alpha_S$
η^*	dimensionless constant = $\omega \cdot r_w^2/\alpha_S$
λ	thermal conductivity, ($W \cdot m^{-1} \cdot K^{-1}$)
μ	constant = δ_n/L, (m^{-1})
θ	transfer function of the quasi-periodical perturbation
ω	angular velocity of fluctuating temperature, ($rad \cdot sec^{-1}$)
ξ	dimensionless position in the slab =x/L
ψ	transfer function of the step perturbation
Θ	temperature difference variable=$T-T_f$, (K)

Subscripts

f	fluid
i	initial
n	n^{th} root
S	slab
T	themocouple

1. INTRODUCTION

In the present study, an inverse heat conduction problem is considered, the Sideways Heat Equation. This is a model of a problem, where the temperatures on both sides of a thick wall are to be determined but where only one side is accessible to measurements. These kinds of problems are encountered in many industrial applications

[1]. This can be the case of some pipes in nuclear reactors [2]; and also of thermal storage in solar energy where the stratification of the temperature of the fluid in the tank is of prime importance [3]. The disturbances due to the presence of thermocouples should then be avoided; which suggests the use of the data measured at the outer surface to estimate the inner temperatures. It may also be the case that locating a measurement device (e.g. a thermocouple) on the surface would disturb the measurements so that an incorrect temperature is recorded. In such cases one is restricted to internal measurements, and from these, the surface temperature is estimated. When surface-mounted thermocouples are used for these temperature measurements, specialized techniques are required to minimize the errors associated with conduction along the lead wires and the junction displacement from the surface. A number of useful analytical relationships utilizing Laplace transforms have since been derived by Keltner and Beck [4] for intrinsic and beaded thermocouples mounted on a thick wall. Unfortunately, the difficulties associated with the inversion of these relationships have precluded the use of the solutions to all but the simplest cases of measured response such as step and ramp temperature changes. Because the temperature changes and the measured response associated with nuclear reactor and solar tank are rarely step or linear functions of time, an expanded solution base is required.
The purpose of this study is therefore to propose Duhamel Integral [5] to include ideal intrinsic thermocouples with a measured response that can be approximated by a quasi-periodical function. Two types of solutions are generally used to solve such problem. In the first type, minimization of the criteria of comparison (the standard deviation) between measured and calculated temperatures is used [6]. However, this approach, besides the fact of being in general time-consuming, suggests that a form of the solution is known before-hand, which is quite difficult to satisfy for all practical cases. The second approach, adopted in the present investigation, is an analytical method based on the transfer function of the solid wall to which an ideal intrinsic thermocouple is attached. This method has been used by Morilhat et al. [2] to determine the parietal temperature fluctuations that occur in the inner surface of some pipes in PWR nuclear reactor. However, their derived transfer function did not take into account the thermocouple effect. In the present work, all the effects are included and the transfer function taken as the product of two transfer functions, one related to the solid and the other to the thermocouple. Both transfer functions are presented separately in dimensionless forms. Discussions about the parameters affecting the response are given and the transfer function of the vapour generator pipe is presented as example. To validate the model, comparison to the measurements performed by Morilhat et al. is given. To the best of our knowledge, no other significant experimental work exists on the subject at hand. It can be shown from the obtained standard deviations between the measured and calculated temperatures, that taking into account the effect of the presence of a thermocouple drastically improves the accuracy of the model.

2. ANALYTICAL CONSIDERATIONS AND HYPOTHESES

The geometry of the problem being considered is shown in Fig. 1. Two homogeneous, isotropic bodies with temperature and space independent thermal properties are in thermal contact over a circular region of radius r_w. The thermocouple is welded to the outer surface of the solid wall; therefore, the interface is considered to have a perfect thermal contact. Except for the contact area, the surface of the semi-infinite body is considered to be adiabatic. This is the case for many industrial applications where a thermal insulation is used. The thermocouple is considered as intrinsic (no protective sheath) and a single semi-infinite cylinder with no lateral heat loss.
The present inverse thermal problem considers as input signal the measured temperature at the outer surface of the pipe, and as output signal the calculated slab inner temperature. At the inner surface of the pipe the coefficient of heat transfer pipe-fluid is considered as infinite, which is a good approximation particularly for metal liquids. Locally, the pipe is considered as a slab (case of a pipe with a thickness negligible comparatively to its outer radius). The conduction heat transfer is considered as one-dimensional (in radial direction for the case of a cylindrical pipe). Any internal thermal non-equilibrium (calorific energy loss by joule effect) or external caused by radiation is negligible. Finally, the time variation of the fluid temperature is assumed quasi-periodical, with a frequency sufficiently low so that the inertia terms in the equation of propagation of heat can be neglected (hypothesis of validation of Fourier equation).

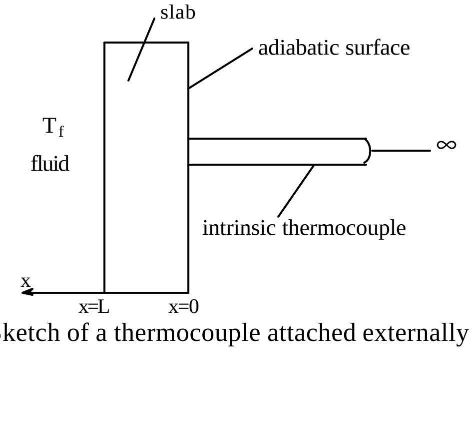

Fig.1. Sketch of a thermocouple attached externally to an adiabatic solid surface

3. DETERMINATION OF THE TRANSFER FUNCTIONS

3.1. Slab transfer functions

3.1.1. Transfer function to the step perturbation

The governing equation is the linearized diffusion equation giben by:

$$\frac{\partial T}{\partial t} = \alpha_S \frac{\partial^2 T}{\partial x^2} \tag{1}$$

where α_S is the thermal diffusivity of the slab.

The slab is considered as semi infinite, of thickness (L). One face is insulated (x = 0) and the other one in contact with a fluid (x = L) with a convection heat transfer coefficient h. Therefore, the initial and boundary conditions are given as:

At time t = 0, $T = T_i$ or $\Theta = T_i - T_f$ (2)

At time t > 0 for x = L, $\left.\begin{array}{l} \frac{\partial T}{\partial x} = -\frac{h}{\lambda_S}(T - T_f) \\ \frac{\partial \Theta}{\partial x} = -\frac{h}{\lambda_S} \end{array}\right\}$ (3)

At time t > 0 for x = 0, $\frac{\partial \Theta}{\partial x} = 0$ (4)

Application of the initial and boundary conditions to Eqs. (2-4) and using the separation of variables method give the response of the slab to a step perturbation of the fluid temperature. This response takes the following dimensionless form:

$$\frac{T - T_i}{T_f - T_i} = 1 - 2.\sum_{n=1}^{\infty} \frac{\sin\delta_n \cos\delta_n \xi}{\delta_n + \sin\delta_n \cos\delta_n} e^{-(\delta_n^2 . NFo)} \tag{5}$$

ξ is the dimensionless position in the slab, NFo is the dimensionless time (Fourier number) defined as: $NFo = \alpha_S \cdot t/L^2$ and $\delta_n = \mu_n L$ is the solution of a transcendant equation given by:

$$\delta_n tg \cdot \delta_n - Bi = 0 \tag{6}$$

Bi being the Biot number. Equation (6) is to be used for the three kinds of boundary conditions at $\xi = 1$ depending on the value of Bi, which can be zero (adiabatic case), moderate (convection case) or infinite. Using the dimensionless parameter $K_n(\xi)$, the transfer function ψ_e can be written as:

$$\psi_S(\xi, NFo) = \frac{T - T_i}{T_f - T_i} = 1 - \sum_{n=1}^{\infty} K_n(\xi) \cdot e^{-(\delta_n^2 . NFo)} \tag{7}$$

with $K_n(\xi)$ being:

$$K_n(\xi) = \frac{2 \cdot \sin\delta_n \cos\delta_n \xi}{\delta_n + \sin\delta_n \cos\delta_n} \tag{8}$$

3.1.2. Transfer function to the quasi-periodical perturbation

The mathematical model developed in the case of a response to a step perturbation will be used as a base to find the solution to the quasi-periodical perturbation case. The solution proposed here is based on Duhamel Integral. Knowing the response of a system to a step perturbation, Duhamel Integral allows us to obtain the response of this system to any kind of perturbation. Using Duhamel's Integral in the case where the fluid temperature is of the following simplified form $T_f(t) = T_i + (\Delta T) \cdot \sin \omega t$, the response of the slab will then have the following form:

$$T(\xi, t) = T_i + (\Delta T) \cdot \left[\sqrt{M_{S2}^2 + (1 - M_{S1})^2}\right] \cdot \sin(\omega t + \phi_S) \tag{9}$$

ϕ_S is the phase lag, and M_{S1} and M_{S2} being:

$$M_{S1} = \sum_{n=1}^{\infty} \frac{K_n(\xi)}{\frac{1}{A_n^2}+1}, \; M_{S2} = \sum_{n=1}^{\infty} \frac{K_n(\xi)}{\frac{1}{A_n}+A_n}, \; \phi_S = \mathrm{Arctg}\left(\frac{M_{S2}}{1-M_{S1}}\right) \tag{10}$$

and

$$A_n = \frac{\omega}{\alpha_S \cdot \mu_n^2} \text{ and } \omega = 2\cdot\pi\cdot f \tag{11}$$

The transfer function of the slab is therefore:

$$\theta_S = \frac{T(\xi, t) - T_i}{\Delta T} = \sqrt{M_{S2}^2 + (1-M_{S1})^2} \cdot \sin(\omega t + \phi_S) \tag{12}$$

Using the dimensionless parameter η defined as: $\eta = \omega \cdot L^2 / \alpha_S$ equation (12) is rewritten:

$$\theta_S = \frac{T(\xi, NFo) - T_i}{\Delta T} = \sqrt{M_{S2}^2 + (1-M_{S1})^2} \cdot \sin(\eta \cdot NFo + \phi_S) \tag{13}$$

From Eqs. (9-13) it can be noticed that the transfer function of the slab depends on:

- The interaction fluid-slab expressed by the parameter d_n or the Biot number Bi.
- The geometry of the slab characterized by the dimensionless parameter $K_n(\xi)$.
- The thickness of the slab L.
- The frequency of perturbations *f*.
- The dimensionless position in the slab ?.
- The thermal diffusivity α_S of the slab.

In Figure 2, the transfer function θ_S of the slab is plotted versus the dimensionless position ξ and the dimensionless angular velocity η. The figure corresponds to a dimensionless time NFo equal 1. It can be noticed that the transfer function decreases from the inner surface of the slab (ξ=1) to the outer position (ξ=0) regardless the value of η, while the phase lag increases. For a given position ξ the amplitude of the transfer function ($\sqrt{M_{S2}^2 + (1-M_{S1})^2}$), decreases with η, while the phase lag ϕ_S increases.

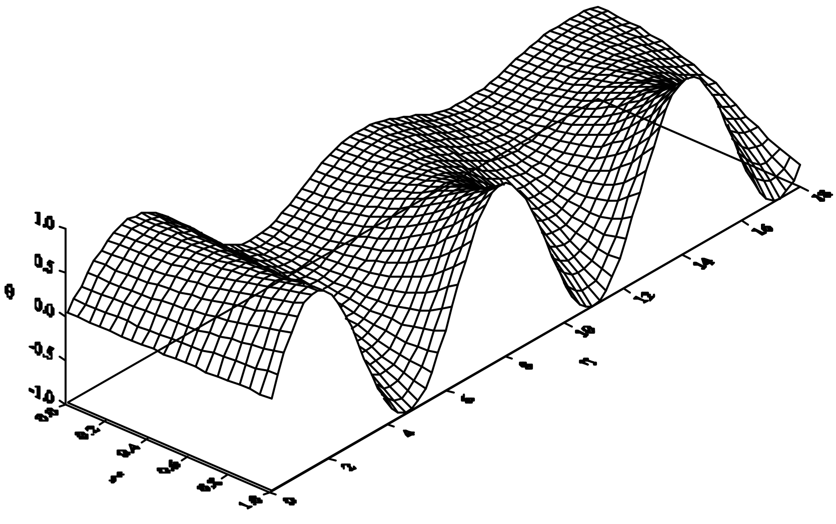

Fig.2. Transfer function for the slab (NFo=1)

3.2. Intrinsic thermocouple transfer functions

3.2.1. Transfer function to the step perturbation

The relationships used for the step response ψ_T of an intrinsic thermocouple with negligible thermal resistance and lateral heat transfer to the surrounding are derived by Keltner and Beck [4]. With t^* being the dimensionless time defined as: $t^* = \alpha_S \cdot t / r_w^2$, the late time solution (t^* >0.1) is given as:

$$\psi_T(t^*) = 1 - B_1 \exp(B_2^2 \cdot t^*)\left[1 - \mathrm{erf}\left(B_2\sqrt{t^*}\right)\right] \tag{14}$$

where

$$B_1 = \beta \Big/ \left(8/\pi^2 + \beta\right) \tag{15}$$

$$B_2 = 4 \Big/ \left(8/\pi + \beta \cdot \pi\right) \tag{16}$$

Keltner and Beck found that the level of the response at zero time and the transient response up to the final value are governed by a single parameter, β.

3.2.2. Transfer function to the quasi-periodical perturbation

Similarly to the slab case, application of Duhamel Integral gives the transfer function to the quasi-periodical perturbation as:

$$\theta_T = \frac{T(t^*) - T_i}{\Delta T} = \sqrt{M_{T2}^2 + (1 - M_{T1})^2} \cdot \sin\left(\eta^* \cdot t^* + \phi_T\right) \tag{17}$$

with $\eta^* = \omega \cdot r_w^2 / \alpha_S$ and M_{T1}, M_{T2} and ϕ_T being:

$$M_{T1} = \frac{B_1\left[\left(\sqrt{\omega}/D\right)^4 + \sqrt{2} \cdot S_2(\omega t) \cdot \left(\sqrt{\omega}/D\right) - \sqrt{2} \cdot C_2(\omega t) \cdot \left(\sqrt{\omega}/D\right)^3\right]}{\left[1 + \left(\sqrt{\omega}/D\right)^4\right]} \tag{18}$$

$$M_{T2} = \frac{B_1\left[\left(\sqrt{\omega}/D\right)^2 - \sqrt{2} \cdot S_2(\omega t) \cdot \left(\sqrt{\omega}/D\right)^3 - \sqrt{2} \cdot C_2(\omega t) \cdot \left(\sqrt{\omega}/D\right)\right]}{\left[1 + \left(\sqrt{\omega}/D\right)^4\right]} \tag{19}$$

$$\phi_T = \mathrm{Arctg}\left(\frac{M_{T2}}{1 - M_{T1}}\right) \tag{20}$$

The parameter D is given as:

$$D = B_2 \cdot \sqrt{\alpha_S / r_w^2} \tag{21}$$

$C_2(\omega t)$ and $S_2(\omega t)$ are the Fresnel integrals [7] defined as:

$$C_2(\omega t) = \frac{1}{\sqrt{2\pi}} \int_0^{\omega t} \frac{\cos x}{\sqrt{x}} dx, \quad S_2(\omega t) = \frac{1}{\sqrt{2\pi}} \int_0^{\omega t} \frac{\sin x}{\sqrt{x}} dx \tag{22}$$

In Figure 3, the transfer function θ_T of the intrinsic thermocouple is plotted versus the dimensionless parameter β and the dimensionless angular velocity η^*.

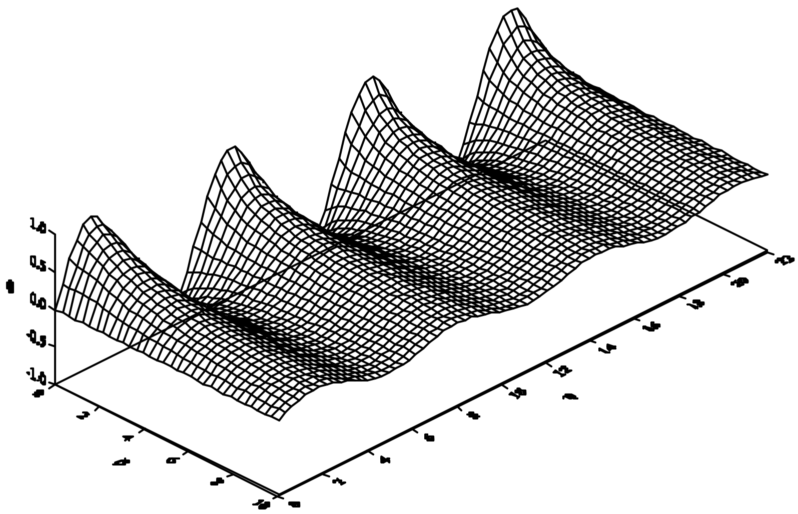

Fig.3. Transfer function for the intrinsic thermocouple ($t^*=1$)

The figure corresponds to a dimensionless time t^* equal 1. It can be noticed that the transfer function depends on a single parameter β, which is similar to the case of step perturbation as found in Keltner and Beck [4]. θ_T deceases with β regardless the value of η^*. The main difference with the slab case concerns the phase lag. While it is important for the slab case, it is almost negligible for intrinsic thermocouple. In addition, for a given parameter β, the dimensionless angular velocity η^* has almost no effect on the transfer function.

4. VALIDATION OF THE RESULTS

To validate the present proposed analytical model, comparison with the experimental results of Morilhat et al.[2] is presented. The experimentation concerns measurements of temperatures of the inner and outer surfaces of a pipe in the vapour generator in the PWR nuclear reactor. The pipe is made of the Carbon Steel A 42 with thermophysical properties λ_S= 42 $W \cdot m^{-1} \cdot K^{-1}$ and α_S = 12,65. 10^{-6} $m^2 \cdot sec^{-1}$ [2], and the thickness L = 21,4 mm. The measurements of temperature were made using thermocouples. However, no details on the type and the size of the used thermocouples were given.

In figure 4, using the present model, the transfer function at the outer surface of vapour generator pipe is plotted versus the frequency (Hz) of temperature variation. The coefficient h of heat transfer pipe-fluid is considered as infinite. It can be noticed that for a frequency above 0.35 Hz, the transfer function is equal zero, which means that no temperature variations can be recorded.

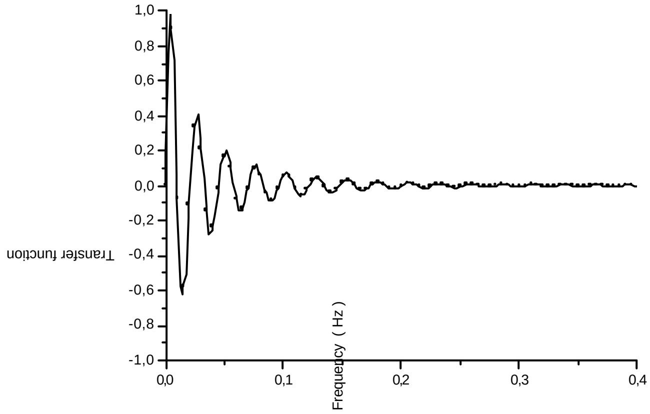

Fig.4. Transfer function of the vapour generator pipe, (ξ=0)

Calculation of the pipe's inner temperature-time history from the measured outer temperature is performed. In order to check the influence of the attached intrinsic thermocouple, an example of type K with λ_T= 19,25 $W \cdot m^{-1} \cdot K^{-1}$ and α_T = 4,92. 10^{-6} $m^2 \cdot sec^{-1}$ [8], and OD=1 mm is assumed. Under this condition, the parameter β is found equal to 0.73. Therefore, the global transfer function is the product of the transfer function of the pipe and the intrinsic thermocouple.

The standard deviation between the measured inner temperatures and the calculated ones is equal 0.12, which can be considered as good for the present problem, taking into account that the actual temperature variations are not exactly quasi periodical as it can be seen in Fig.5, and that the thermal contact between the intrinsic thermocouple and the pipe is considered as perfect, which is a little bit different in actual situation.

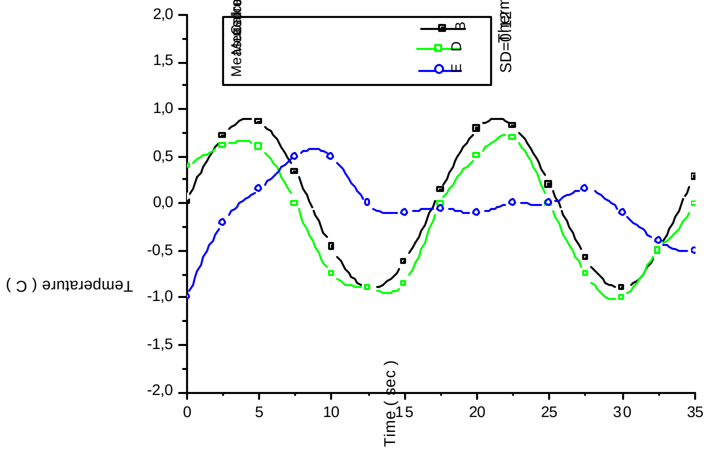

Fig.5. Calculated and measured temperatures in the vapour generator pipe

If the effect of the presence of a thermocouple is neglected in the model as it was the case in the model proposed by Morilhat et al. [2], the standard deviation is equal 0.15. This suggests that the present model predicts more accurately the inner temperature variations in the pipe. For the thermal striping phenomena that can occur inside the pipe due to the temperature variation and lead to its damage, its is not possible to predict it for the present type of the pipe (thermophysical properties and dimensions) and a frequency range above 0.35 Hz as shown in Fig.4.

CONCLUSION

In the present study, a solution of the Sideway Heat Equation is given. It is based on Duhamel Integral. Taking into account the effect of the presence of the attached thermocouple, the transfer function is given. Application to an actual problem of a pipe in the vapour generator of a PWR Nuclear Reactor, shows that the present model predicts more accurately the inner temperature variations than the previous model proposed by Morilhat et al. [2]. However, the prediction of the damage due to the thermal striping phenomena that occur inside the pipe, and for the present studied type of the pipe and a frequency above 0.35 Hz, it is not possible to solve the problem using thermocouples. This was also found in Morilhat et al. [2]. In addition, since it is an inverse heat conduction method, it is expected that the analytical model given herein, can estimate more accurately the thermal diffusivity of solids than the previous methods ignoring in their estimations the presence of thermocouples.

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